Věta 18: Nechť f je fce d proměnných, $I \subseteq \mathbb{R}^d$ je ot. interval $\left(I = \underbrace{I_1 \times I_2 \times \dots \times I_d}_{\text{ot. intervaly v } \mathbb{R}} \subseteq \mathbb{R}^d \right)$

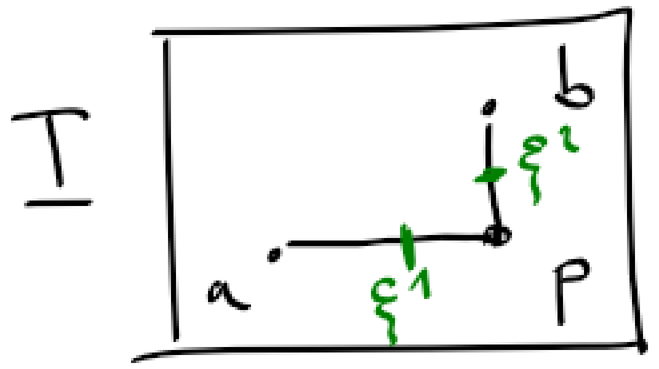
Nechť f má v I všechny PD 1. řádu.

Nechť $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in I$.

Pak existují body $\xi^1, \xi^2, \dots, \xi^d \in I$, že

$$f(b) - f(a) = \sum_{i=1}^{d} (b_i - a_i) \frac{\partial f}{\partial x_i}(\xi^i)$$

Pozn.: • $d = 1$: $I \subseteq \mathbb{R}$, $a, b \in I$;



$$\exists \xi^1 \in I: f(b) - f(a) = (b - a) \cdot f'(\xi^1)$$

• $d = 2$: $I = I_1 \times I_2 \subseteq \mathbb{R}^2$ (obdélník), $a, b \in I$.

$$\exists \xi^1, \xi^2 \in I: f(b) - f(a) = (b_1 - a_1) \frac{\partial f}{\partial x_1}(\xi^1) + (b_2 - a_2) \frac{\partial f}{\partial x_2}(\xi^2)$$

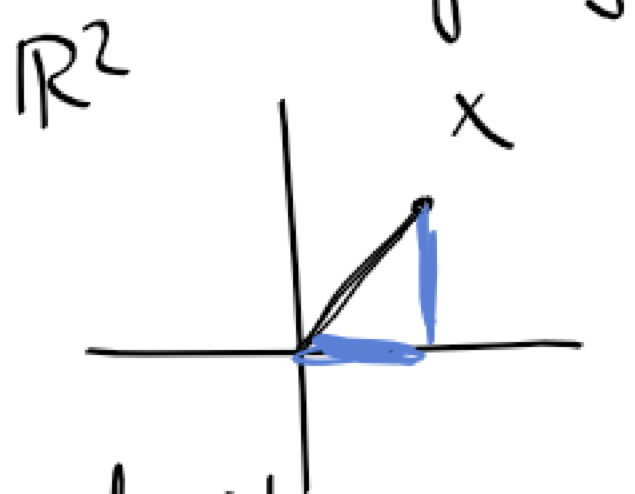
$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_d^2}$$

Věta 19: Nechť jsou všechny P.D. $\frac{\partial f}{\partial x_i}$, $i = 1, \dots, d$ spojité v bodě $a \in \mathbb{R}^d$.

Pak f má v bodě a totální diferenciál $df(a)$.

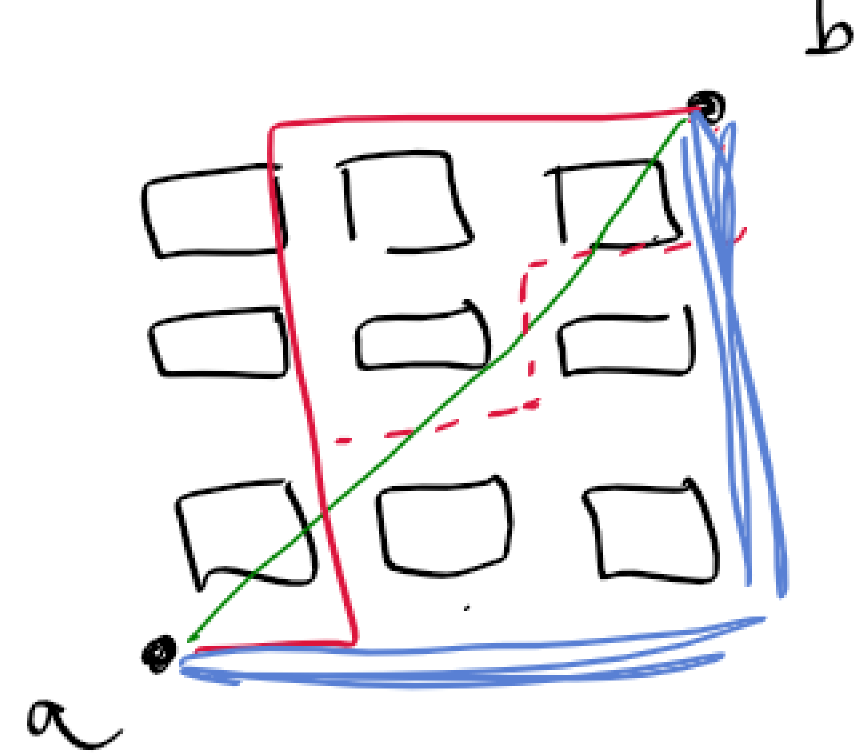
Definice 20: $x \in \mathbb{R}^d$, $x = (x_1, \dots, x_d)$. Definujeme

• $\|x\|_s = \|x\|_1 = \sum_{i=1}^{d} |x_i|$



$\rho_1(x, y) = \|x - y\|_1$ ... NY-metrika

• $\|x\|_{max} = \|x\|_\infty = \max\{|x_i| : i = 1, \dots, d\}$

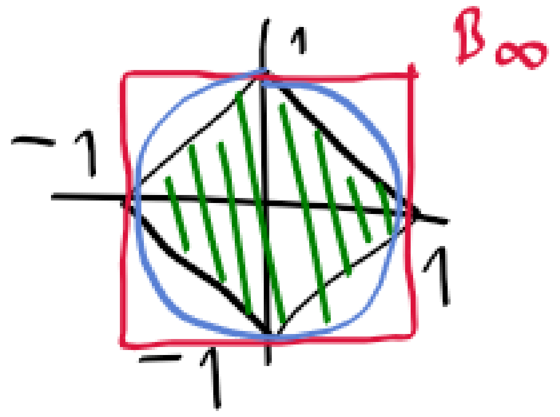


Poznámka:

$B_1(a, r) = B_{\|\cdot\|_1}(a, r) = \{x \in \mathbb{R}^d : \|x - a\|_1 < r\}$

$B_\infty(a, r) = B_{\|\cdot\|_\infty}(a, r) = \{x \in \mathbb{R}^d : \|x - a\|_\infty < r\}$

• v $\mathbb{R}^2$: $B_1((0,0),1)$:



$B_\infty((0,0),1)$

$B_2((0,0),1)$

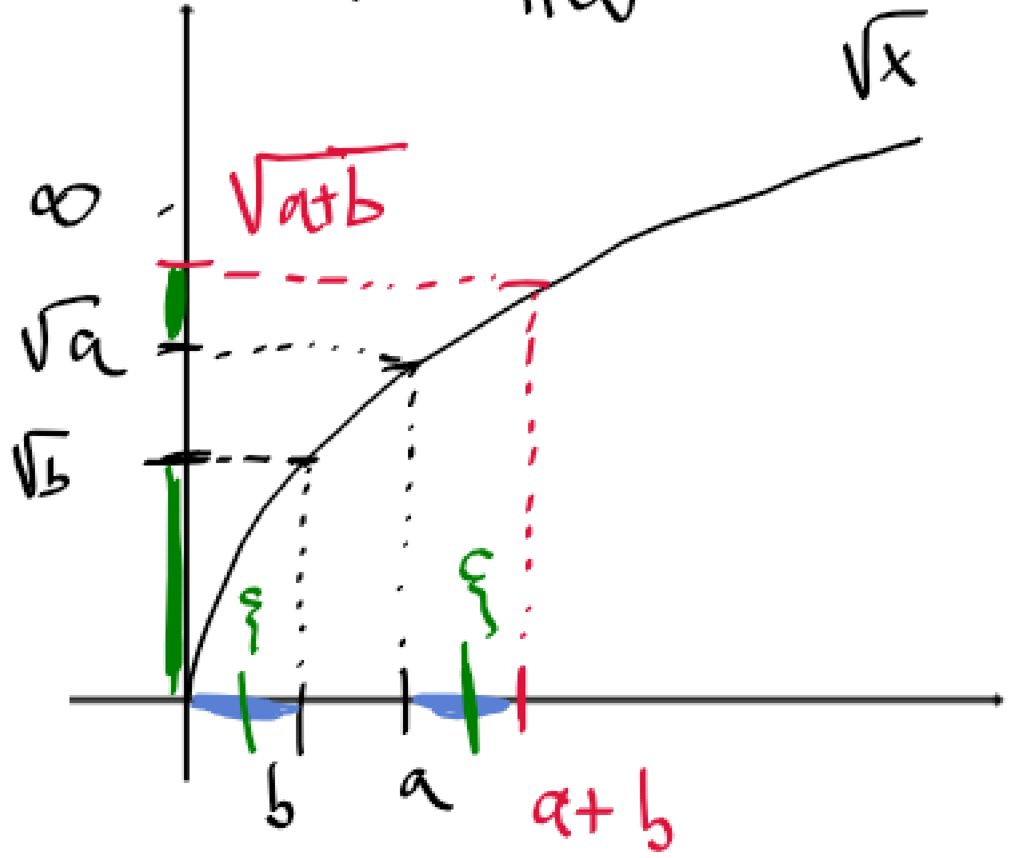
$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}} \quad \text{(pro zajímavost)}$$

Lemma 21: (Ekvivalence norem: $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$)

Pro libovolné $x \in \mathbb{R}^d$, $x = (x_1, \dots, x_d)$ platí:

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le d \cdot \|x\|_\infty$$

navíc: $\|x\|_2 \le \sqrt{d} \cdot \|x\|_\infty$.



Důkaz: nechť $x \in \mathbb{R}^d$, nechť

$$\|x\|_\infty = \max\{|x_i| : i = 1, \dots, d\} = |x_j|.$$

Pak

$$\|x\|_\infty = |x_j| = \sqrt{x_j^2} \le \sqrt{\sum_{i=1}^{d} x_i^2} \left(= \|x\|_2\right) \overset{(*)}{\le} \sum_{i=1}^{d} |x_i| \le \sum_{i=1}^{d} |x_j| = d \cdot |x_j| = d \cdot \|x\|_\infty.$$

navíc:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{d} x_i^2} \le \sqrt{\sum_{i=1}^{d} x_j^2} = \sqrt{d \cdot x_j^2} = \sqrt{d} \cdot \sqrt{x_j^2} = \sqrt{d} \cdot |x_j| = \sqrt{d} \cdot \|x\|_\infty$$

(*): $d = 2$: $\sqrt{x_1^2 + x_2^2} \overset{?}{\le} \sqrt{x_1^2} + \sqrt{x_2^2} = |x_1| + |x_2|$

Tj. CHCEME: $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ $(a, b \ge 0)$.

Vidět z obr.:

$$\sqrt{a+b} - \sqrt{a} \le \sqrt{b} - \sqrt{0}\ ?$$

$$\frac{\sqrt{a+b} - \sqrt{a}}{b} \le \frac{\sqrt{b} - \sqrt{a}}{b}\ ?$$

Cv.

Důsledek 22: Spojitost a limita vyjdou nastejno pro všechny 3 normy.

Náznak Dk: Například:

$$\lim_{x\to a}{}_{\|\cdot\|_2} f(x) = A \iff \lim_{x\to a}{}_{\|\cdot\|_1} f(x) = A.$$

Dk: $(\Rightarrow)$: MÁME: CHCEME:

$\forall \varepsilon>0\ \exists \delta>0\ \forall x: \|x-a\|_2<\delta \Rightarrow |f(x)-A|<\varepsilon$

CHCEME: MÁME:

$\forall \varepsilon>0\ \exists \delta'>0\ \forall x: \|x-a\|_1<\delta' \Rightarrow$ — " —

$\varepsilon>0$ dáno. $\exists \delta>0\ \forall x: \|x-a\|_2<\delta \Rightarrow$ — " —

Chceme najít $\delta'>0$.

$\|x-a\|_1 \leq d\cdot\|x-a\|_\infty \leqslant d\,\|x-a\|_2$

$\|x-a\|_2 \leqslant \|x-a\|_1$ Podle 21.

Položme ~~$\delta' := \frac{\delta}{d}$~~. Pak:

Pokud $\|x-a\|_1<\delta$, pak

$\|x-a\|_2 \leqslant \|x-a\|_1 < \delta$, a tedy

podle „MÁME": $|f(x)-A|<\varepsilon$.

$(\Leftarrow)$: CHCEME $\longleftrightarrow$ MÁME (naopak).

Tj. nechť $\exists \delta'\ \forall x: \|x-a\|_1<\delta' \Rightarrow$ — " —

Položme $\delta = \frac{\delta'}{d}$. Pokud $\|x-a\|_2<\delta$,

tak $\|x-a\|_1 \leq d\cdot\|x-a\|_2 < d\cdot\delta =$

$= d\cdot\frac{\delta'}{d} = \delta'$. Tedy

$|f(x)-A|<\varepsilon.$ ⊠

Důkaz V19: jediný kandidát na TD je

$$L(h) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(a) \cdot h_i \quad \text{, tj. } L = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_d}(a)\right)$$

Chceme dokázat, že L je TD, tj.:

$$\lim_{h \to 0} \frac{\overbrace{f(a+h) - f(a)}^{f(b)-f(a)} - \left(\frac{\partial f}{\partial x_1}(a) h_1 + \dots + \frac{\partial f}{\partial x_d}(a) h_d\right)}{\|h\|_2}$$

$$= 0 = \lim_{h \to 0} \frac{\eta(h)}{\|h\|} \qquad \left(\text{Tj. } \eta(h) = o(\|h\|),\ h \to 0\right)$$

Zvolme $\varepsilon > 0$: najdeme $\delta > 0$, že

$$\forall x \in I := (a_1 - \delta, a_1 + \delta) \times (a_2 - \delta, a_2 + \delta) \times \dots \times (a_d - \delta, a_d + \delta) =$$

$$= B_\infty(a, \delta). \qquad \left[\delta = \min\{\delta_1, \dots, \delta_d\}\right]$$

$$\left|\frac{\partial f}{\partial x_1}(x) - \frac{\partial f}{\partial x_1}(a)\right| < \frac{\varepsilon}{d}, \dots, \left|\frac{\partial f}{\partial x_d}(x) - \frac{\partial f}{\partial x_d}(a)\right| < \frac{\varepsilon}{d}.$$

(díky spojitosti $\frac{\partial f}{\partial x_i}$ v bodě a, $i = 1, \dots, d$)

Nechť $h \in \boxed{P}_\infty(0, \delta)$, tj. $\boxed{0} < \|h\|_\infty < \delta$.

Označme $b = a + h \in I = B_\infty(a, \delta)$.

Podle V18: $\exists \xi^1, \dots, \xi^d \in I$:

$$f(b) - f(a) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(\xi^i) \cdot (b_i - a_i). \text{ Tedy}$$

$$|\eta(h)| = \left|\sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(\xi^i) \cdot \underbrace{(b_i - a_i)}_{h_i} - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(a) \cdot \underbrace{(b_i - a_i)}_{h_i}\right|$$

$$= \left|\sum_{i=1}^{d} h_i \cdot \underbrace{\left(\frac{\partial f}{\partial x_i}(\xi^i) - \frac{\partial f}{\partial x_i}(a)\right)}\right| \leq$$

$$\leq \sum_{i=1}^{d} |h_i| \cdot \left|\frac{\partial f}{\partial x_i}(\xi^i) - \frac{\partial f}{\partial x_i}(a)\right| \overset{\xi^i \in I}{<} \sum_{i=1}^{d} |h_i| \cdot \frac{\varepsilon}{d}$$

$$= \frac{\varepsilon}{d} \sum_{i=1}^{d} |h_i| = \frac{\varepsilon}{d} \|h\|_1.$$

Tedy $\dfrac{|\eta(h)|}{\|h\|_2} \overset{L27}{\leq} \dfrac{|\eta(h)|}{\frac{1}{d}\|h\|_1} = \dfrac{d}{\|h\|_1} \cdot |\eta(h)| <$

[L27: $\|h\|_1 \leq d \cdot \|h\|_2$, tj. $\|h\|_2 \geq \frac{1}{d}\|h\|_1$]

$< \dfrac{d}{\|h\|_1} \cdot \dfrac{\varepsilon}{d} \cdot \|h\|_1 = \varepsilon.$ $\boxtimes$

Poznámka 23: • V16 $\Rightarrow$ $df(a)$ je určen jednoznačně.

• uvažujme násl. výroky o funkci f:

(A) f je spoj. v bodě a

(B) ex. parc. der. (1. ř.) v bodě a

(C) ex. $df(a)$

(D) $\dfrac{\partial f}{\partial x_i}$ jsou spoj. v bodě a

Platí implikace
(C) $\Rightarrow$ (B) V16
(C) $\Rightarrow$ (A) V17
(D) $\Rightarrow$ (C) V19

$$\begin{tikzcd} & (D) \arrow[d, Rightarrow] & \\ & (C) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ (A) & & (B) \end{tikzcd}$$

Neplatí žádná jiná implikace mezi těmito 4 výroky.

• (A) $\not\Rightarrow$ (B) $f(x,y) = |x|$

• (B) $\not\Rightarrow$ (A) $f(x,y) = \operatorname{sgn} x \cdot \operatorname{sgn} y$, $a = (0,0)$